ON THE RANK OF A SYMMETRICAL MATRIX.*

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1. A matrix (a_{ij}) is said to be of rank r if some minor M_r of order r is not zero, while every minor M_{r+1} of order $r + 1$ is zero. A minor of order r in which there occur exactly k elements a_{ii} of the main diagonal of a square matrix (a_{ij}) shall be designated by $M_r^{(k)}$. In particular, any $M_r^{(r)}$ is called a principal minor.

Of various theorems† on the rank of a symmetrical matrix, the following theorem ‡ due to Kronecker is especially useful:

In a symmetrical matrix of rank $r (r > 0)$, at least one principal minor of order r is not zero.

The following proof rests upon the fact that if n linear homogeneous equations in n variables have a set of solutions not all zero, the determinant of the coefficients is zero.

2. THEOREM. *If, in a symmetrical matrix (a_{ij}), every $M_{r+1} = 0$ and every $M_r^{(r)} = 0$, then every $M_r = 0$.*

To proceed by induction, let ρ be a fixed integer, $0 \leqq \rho < r$, for which every $M_r^{(k)} = 0$, $k > \rho$. We shall prove that every $M_r^{(\rho)} = 0$. The assumption that there is a non-vanishing $M_r^{(\rho)}$ will be shown to involve a contradiction. After a rearrangement of the rows of (a_{ij}) and the like rearrangement of the corresponding columns, a change not affecting our hypotheses, we may assume that

$$M = |a_{i1} \cdots a_{i\rho} a_{ir+1} \cdots a_{i2r-\rho}| \neq 0 \qquad (i = 1, \cdots, r).$$

Let t be one of the integers $r + 1, \cdots, 2r - \rho$. Then M is a minor of

$$M_{r+1} = |a_{i1} \cdots a_{i\rho} a_{i\rho+1} a_{ir+1} \cdots a_{i2r-\rho}| = 0 \quad (i = 1, \cdots, r, t).$$

Expand the latter determinant according to the elements of its last row (given by $i = t$). In the co-factor of a_{tj} $(j \geqq r + 1)$ appear the elements $a_{kk} (k = 1, \cdots, \rho + 1)$, so that these co-factors are vanishing

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† Cf. Bôcher's Introduction to Higher Algebra, pp. 56–59.

‡ A short proof of different nature may be found in G. Kowalewski's Determinantentheorie, p. 122–4.

$M_r^{(\rho+1)}$. Call A_j the co-factor of a_{tj} $(j \leqq \rho)$. Since the co-factor of $a_{t_{\rho+1}}$ is $\pm M$,

$$a_{t1}A_1 + \cdots + a_{t_\rho}A_\rho \pm a_{t_{\rho+1}}M = 0 \quad (t=r+1, \cdots, 2r-\rho).$$

A like equation with $t \leqq r$ holds since the sum of the products of the elements of a row by the co-factors of the elements of a different row is zero. Since $M \neq 0$, it follows (end of § 1) that

$$|a_{t1} \cdots a_{t_{\rho+1}}| = 0$$

for any $\rho + 1$ values $\leqq 2r - \rho$ of t. Interchange rows with columns and note that $a_{ij} = a_{ji}$. Thus

$$|a_{it}| = 0 \quad (i = 1, \cdots, \rho + 1;\ t \text{ with any } \rho + 1 \text{ values}).$$

Hence every $M_{\rho+1}$ formed from the first $\rho + 1$ rows of M is zero, in contradiction with $M \neq 0$. Thus every $M_r^{(\rho)} = 0$ and the induction is complete.

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